Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N\times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Remark:
- Images are matrices (mathematically)
- For the ease of discussion, we assume $\mathcal{I} = M_{N\times N}(\mathbb{R})$

  (collection of all $N \times N$ real matrices)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

Image processing v.s. Image Transformation

Let g be a "bad" image, which is a distorted version of a good (clean) image f. Then: we can write $g = \mathcal{O}(f)$, where $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ transforms one image to another.

To solve the imaging problem:

(1) Design a suitable $T: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ such that:

$$T(g) = f \quad (\text{i.e. } T \approx \mathcal{O}^{-1})$$

(2) Design mathematical method to solve:

$$g = \mathcal{O}(\underbrace{f}_{\text{unknown}}) \quad (\text{Given } g \text{ and } \mathcal{O}, \text{ we solve for the unknown } f)$$

Definition: (Linear image transformation)

An image transformation $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear if it satisfies:

$$\mathcal{O}(af + bg) = a\,\mathcal{O}(f) + b\,\mathcal{O}(g) \text{ for all } f, g \in M_{N\times N}(\mathbb{R}),\ a, b \in \mathbb{R}.$$

<u>Examples:</u>

- Given $A \in M_{N\times N}(\mathbb{R})$. Define: $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ by: $\mathcal{O}(f) = 2f + Af$ for all $f \in M_{N\times N}(\mathbb{R})$.
  Then: $\mathcal{O}$ is linear.
- Given $A, B \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{O}$ by:
  $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$. $\mathcal{O}$ is linear
- Given $A \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{O}$ by:

  $\mathcal{O}(f) = fAf$. Is $\mathcal{O}$ linear??

Point Spread Function

Take $f \in \mathcal{I} = M_{N\times N}(\mathbb{R})$.

$$\text{Let } f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(x,y) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{x=1}^{N}\sum_{y=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(x,y) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y) \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

(x, y) entry

Consider a linear image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$.

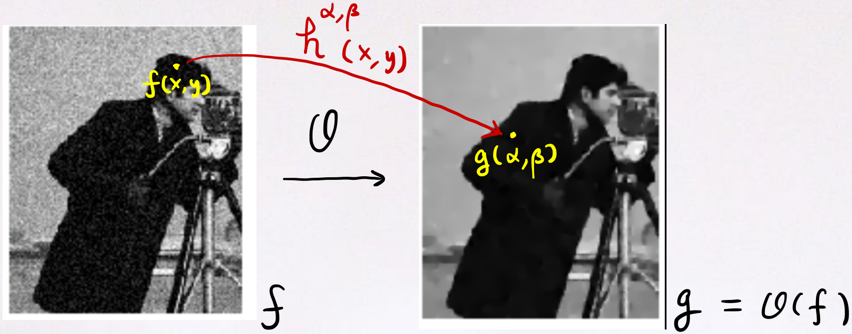
Let $g = \mathcal{O}(f)$. Then:

$$g(\alpha, \beta) = \left[\sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, \mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \right) \right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h^{\alpha,\beta}(x,y) \quad \text{where}$$

$$h^{\alpha,\beta}(x,y) = \left[\mathcal{O}(P_{xy}) \right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{cc} & y^{th} \\ & \downarrow \\ & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \leftarrow x^{th} \end{array}$$

Remark: $h^{\alpha,\beta}(x,y)$ determines how much the pixel value of f at (x,y) influences the pixel value of g at (α,β).



Definition: (Point spread function)

$h^{\alpha,\beta}(x,y)$ is usually called the point spread function (PSF)